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A case of indispensable localized instability in elastic–plastic solids

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Abstract

The primary instability in homogeneous elastic-plastic bodies with prescribed boundary displacements is studied. In this event instability is known to arise when Hadamard's inequality is first violated, this violation being the very condition for the localized instability to be possible in principle. The question is posed whether localized instability is the only possible type of instability under specified conditions, or diffuse instability is equally possible (which is true for elastic bodies).

In order to state a rational criterion for distinction between localized and diffuse instability modes (IMs) (which are treated in full generality as mutually complementary notions without any a priori prescriptions regarding the mode of deformation), it is proposed to characterize IMs by means of some quantitative measure of localization named the 'localizational volume'. The latter evaluates the volume of that part of a body, where relatively great incremental strains are concentrated (this property of proposed measure is proved).

The main result established is that in the problem under consideration any primary IM is characterized by infinitesimal value of localizational volume, i.e. all the primary IMs appear to be localized in such a 'volumetric' sense, which means at least the absence of diffuse IMs.

The conclusion is drawn that indispensability of such a localization (treated in the sense of small localizational volume) is a global, essentially non-linear effect (boundary constraint + piecewise − linear constitutive relation).

1. Introduction

It is well-known from observations in the fields of geology, technology and material testing, that the zones of strain localization are formed at a certain stage of deformation process, such a behaviour being typical for a variety of materials ranging from soils and rocks to metals.

Study of the localization phenomena, apart from its evident importance for a good number of

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applied disciplines, is also of interest from a purely theoretical standpoint, since localization is conventionally considered as a mechanism of incipiency of discontinuities in initially continuous medium.

Such a concept of localizational incipiency of discontinuities was stated in clear and complete form by Rudnicki and Rice (1975) and Rice (1977) on the grounds of fundamental results of Hill (1962), based on the classical Hadamard's investigations on elastic stability. The concept has got an extremely wide acknowledgement and spread and led to a great number of researchers attacking the problem of localization analytically as well as numerically, using the ideas and methods of foregoers, often in roughly simplified form.

As for the grounds, the classical paper of Hill (1962) contains a draft of proof of the basic Hadamard stability theorem for the case of elastic-plastic solids (the original theorem concerns purely elastic ones). From the theorem together with the proof it follows firstly, that instability in elastic-plastic body arises no later than Hadamard's inequality is violated (for the moduli tensor of plastic response), and secondly, that peculiar localized IMs (the 'Lüders band' distortions, according to Hill's terminology) are possible as the modes of manifestation of that type of constitutive degeneracy.

In Rudnicki and Rice (1975) and Rice (1977) this result is employed and specified for an elastic-plastic constitutive relation, that describes the behaviour of soils and rocks, and the above-mentioned concept is stated.

From what is proven in all mentioned basic works on localization, it by no means follows that the localized instability is the only possible or in any sense preferable type of instability under some kind of conditions. Nevertheless, in the majority of papers which follow Rudnicki and Rice (1975) and Rice (1977) and where their method of analysis is used, such an idea is regarded as self-evident and requiring no substantiation. Such a conviction results probably from the fact that the actual experimental data confirm preference of the localized instability, and thus, it seems to be already explained theoretically (within the framework of the model employed). However, in general it cannot be proved, since under the conditions when active plastic loading is possible all over the body and hence, the response of elastic-plastic body is the same as that of some hypothetical (so-called 'comparison') elastic body, diffuse IMs are possible along with localized ones (see the example in Section 3).

Taking all of this into account, it seems desirable (in order to render the concept of localizational incipiency of discontinuities, more complete and convincing) really to prove theoretically the indispensability of just localized primary instability, at least for some restricted cases.

It should be pointed out that somewhat different manners of the onset of localization are observed in the experiments of different types; the variety of known facts is discussed in detail by Rice (1977). From all discussed there, it is possible to single out two basic variants. The first one is characterized by diffuse primary instability, whereas the localized mode is formed during further post-critical deformation. Theoretical analysis of such a type of evolution of deformation process for biaxially stretched sheets was proposed by Petryk and Thermann (1996). The second one is characterized by the fact that the localized pattern emerges from a preceding uniform state instantly and at once as a whole. It seems natural to treat it as a localized primary IM.

An attempt to explain theoretically the second of the observed types of the onset of localization, i.e., to prove that under certain conditions only localized primary IMs are possible, is exactly the purpose of the paper, that continues previous investigations of the author (Ryzhak, 1993) in the same direction.

In the context of this study, it appeared natural to give up the very tight notion of localization as concentration of relatively great strains within extremely thin plane layers (the Lüders band IMs). Thus, the shape of a region of strain concentration and strain distribution within it are not specified a priori, whereas localized and diffuse types of deformation are understood in full generality as mutually complementary types, embracing in couple the whole range of possible deformation modes.

With such a broad treatment of localization, the only way to distinguish between localized and diffuse IMs, is to state some rational criterion based upon a quantitative measure of localization, that would characterize each IM. It is proposed to use for this purpose a quantity of 'localizational volume' (arising in the course of instability analysis) which appears to evaluate the volume of a part of a body, where relatively great incremental strains corresponding to an IM, are concentrated (this property is proved). The IM is said to be localized if its localizational volume is small as compared with the whole volume of a body.

The main result established is that in homogeneous elastic-plastic bodies under the displacement boundary conditions, all the primary IMs are characterized by infinitesimal values of the localizational volume. This does not mean that they are of shear (Lüders) band type, but that they are at least not diffuse, which adds some needed significant element to the concept of localizational mechanism of incipiency of discontinuities in solids.

From the argument it is clear that such a 'volumetric' localization is in principle a non-linear global effect caused by incremental piecewise linearity of elastic-plastic constitutive relation (two sets of the moduli) and by the influence of a boundary constraint, which makes both sets manifest themselves.

Note that violation of Hadamard's inequality is necessary for the localized type of instability to occur. Hence, if instability arises before violation of Hadamard's inequality, then it is surely diffuse, e.g. when the traction boundary conditions are posed on a part of the boundary (Hutchinson and Miles, 1974; Miles, 1975). Thus, for the localized IMs to be possible in principle, stability must be preserved up to violation of Hadamard's inequality. The most typical conditions for it (Hill, 1962) are the homogeneity of a body together with rigid boundary constraint (the case just considered here). However, stability can be preserved up to the same critical stage under different conditions—some of the cases are considered by Nikitin and Ryzhak (1986) and Ryzhak (1993a, b, 1994), the main result of the paper still remaining valid (see some of the examples in Section 5).

Gibbs' system of tensor notation, supplemented with the tensor product sign, is used throughout; summation convention is never employed.

2. Constitutive relations

A material is supposed to be elastic-plastic, obeying the following incremental piecewise-linear associative constitutive relation (Hill, 1958, 1959):

$$\delta \mathbf{T}_{\varkappa} = \begin{cases} \mathbf{C}^{\mathrm{p}}:\delta\mathbf{H}, & \delta\mathbf{H}:\mathbf{S} \geqslant 0 \\ \mathbf{C}^{\mathrm{e}}:\delta\mathbf{H}, & \delta\mathbf{H}:\mathbf{S} \leqslant 0 \end{cases},$$

$$\mathbf{C}^{\mathrm{p}} = \mathbf{C}^{\mathrm{e}} - \theta \mathbf{S} \otimes \mathbf{S}, \quad \theta > 0, \tag{1}$$

where $\delta\mathbf{H} \equiv \nabla \otimes \delta\mathbf{u}(\mathbf{x})$ is the infinitesimal distortion tensor that corresponds to the field of infinitesimal displacements $\delta\mathbf{u}(\mathbf{x})$ with respect to current configuration $\varkappa$ (taken as a reference configuration), $\delta\mathbf{T}_{\varkappa}(\mathbf{x})$ is a corresponding increment of the Piola stress tensor at a certain material point, $\mathbf{C}^{e}$ and $\mathbf{C}^{p}$ are the moduli tensors for elastic and plastic responses, and $\mathbf{S}$ is a symmetric second-order tensor that specifies the normal to the smooth yield surface in distortion space. Positiveness of θ means that a material is supposed to be stiffer under elastic unloading than under active plastic loading (the latter is a conventional supposition).

3. Stability and instability criteria—the primary IMs

We shall examine stability and instability of uniform equilibrium states of homogeneous elastic-plastic bodies, under the displacement boundary conditions. Consider some family $\varkappa(q)$ of the equilibrium configurations of that type versus a parameter q and suppose that there is some threshold value q_*, i.e. for $q < q_*$ the states are stable and for $q > q_*$ they are unstable.

It is implied that the body undergoes some quasi-static uniform deformation specified by the prescribed infinitely slow motion of material points of the boundary, and in the course of this process the parameter q increases monotonically. Thus, instability arises first at $q = q_* + 0$ and the primary instability is just the matter studied here (without any customary preliminary suppositions regarding its modes).

Whatever definition of stability or instability we make use of (Hill, 1958, 1959, 1978; Drucker, 1959, 1964), we arrive at the same mathematical criterion of presence (for stability) or absence (for instability) of positive definiteness of the functional

$$R\{\delta\mathbf{u}\} \equiv \int_{\varkappa} \delta\mathbf{H}{:}\mathbf{C}{:}\delta\mathbf{H}\,\mathrm{d}V - \int_{\partial\varkappa} \delta\mathbf{t}_{\varkappa} \cdot \delta\mathbf{u}\,\mathrm{d}\Sigma \tag{2}$$

versus the admissible infinitesimal virtual displacement fields $\delta\mathbf{u}(\mathbf{x})$ with respect to equilibrium configuration $\varkappa(q)$ taken as a reference one. Here $\mathbf{t}_{\varkappa}$ is the Piola boundary traction and the tensor $\mathbf{C}$ is meant according to eqn (1) to take the values of $\mathbf{C}^{p}$ or $\mathbf{C}^{e}$ dependent on the sign of the product $\delta\mathbf{H}:\mathbf{S}$; the tensor fields $\mathbf{C}^{p}(\mathbf{x})$, $\mathbf{C}^{e}(\mathbf{x})$ and $\mathbf{S}(\mathbf{x})$ are constant in $\mathbf{x}$ (homogeneity of a body and uniformity of quasi-static process of its deformation). The fields $\delta\mathbf{u}(\mathbf{x})$ are supposed to be continuous, piecewise-smooth and vanishing over the boundary (displacement boundary conditions). Then the surface integral in eqn (2) vanishes:

$$R\{\delta\mathbf{u}\} = \int_{\varkappa} \delta\mathbf{H}{:}\mathbf{C}{:}\delta\mathbf{H}\,\mathrm{d}V. \tag{3}$$

We exclude indifferent equilibrium (when $R\{\delta\mathbf{u}\}$ is positive semi-definite) from instability, i.e. consider as unstable only the states for which $R\{\delta\mathbf{u}\}$ takes the negative values, and any admissible displacement field giving a negative value to the functional we call the IM. It should be mentioned, however, that the states of indifferent equilibrium in the case considered are absent (proved below).

We introduce the Hadamard number c of the fourth-order tensor $\mathbf{C}$ as follows:

$$c \equiv \min_{|\mathbf{f}|=|\mathbf{g}|=1} \mathbf{f}\otimes\mathbf{g}{:}\mathbf{C}{:}\,\mathbf{f}\otimes\mathbf{g}, \tag{4}$$

$$c^{\mathrm{p}} \leqslant c^{\mathrm{e}}. \tag{5}$$

Then Hadamard's condition and that of strong ellipticity are equivalent to non-negativeness and strict positiveness of the Hadamard number, respectively.

If $c^{\mathrm{p}} < 0$, then it follows from the basic Hadamard stability theorem, extended onto the case of elastic-plastic bodies (Hill, 1962; Ryzhak, 1987; Petryk, 1992), that the body is unstable, and IMs constructed in the proofs are clearly localized (of shear band type).

On the other hand, using the inequality

$$\forall \delta\mathbf{H}, \quad \delta\mathbf{H}{:}\mathbf{C}^{\mathrm{p}}{:}\delta\mathbf{H} \leqslant \delta\mathbf{H}{:}\mathbf{C}^{\mathrm{e}}{:}\delta\mathbf{H} \tag{6}$$

we get the following minorization:

$$R\{\delta\mathbf{u}\} \geqslant R^{\mathrm{p}}\{\delta\mathbf{u}\} \equiv \int_{\varkappa} \delta\mathbf{H}{:}\mathbf{C}^{\mathrm{p}}{:}\delta\mathbf{H}\,\mathrm{d}V. \tag{7}$$

If $c^{\mathrm{p}} \geqslant 0$, then due to theorem of Van Hove (1947), some additional inequality holds for the minorizing functional:

$$R^{\mathrm{p}}\{\delta\mathbf{u}\} \geqslant c^{\mathrm{p}} \int_{\varkappa} \delta\mathbf{H}{:}\delta\mathbf{H}\,\mathrm{d}V \geqslant 0, \tag{8}$$

which means that instability arises here only when c^{p} becomes negative. Hence, if $c^{\mathrm{p}}(q)$ is supposed to decrease in q, then the primary IMs correspond to the value $c^{\mathrm{p}} = -0$ and hence, the localized IMs occur among them.

However, for an incrementally linear homogeneous body (i.e. possessing only one set of the tangent moduli) the diffuse IMs are equally possible under the same conditions, and the following example serves to illustrate this assertion.

Consider a homogeneous elastic body characterized by linear incremental constitutive relation

$$\delta\mathbf{T}_{\varkappa} = \mathbf{C}{:}\delta\mathbf{H}, \tag{9}$$

where the moduli tensor $\mathbf{C}$ is constant in $\mathbf{x}$. Let the Hadamard number c be negative and correspond to a pair of unit vectors $\mathbf{n}_0$, $\mathbf{g}_0$:

$$c = \mathbf{n}_0 \otimes \mathbf{g}_0{:}\mathbf{C}{:}\mathbf{n}_0 \otimes \mathbf{g}_0 < 0. \tag{10}$$

Let the rectangular parallelepiped

$$0 \leqslant \mathbf{x}\cdot\mathbf{e}_i \leqslant l_i, \quad i = 1,2,3, \mathbf{e}_i\cdot\mathbf{e}_j = \delta_{ij}, \quad \mathbf{e}_1 = \mathbf{n}_0,$$

lie within the body. We set the following sequence of incremental displacement fields vanishing outside the parallelepiped:

$$\delta\mathbf{u}_n(\mathbf{x}) = \mathbf{g}_0 \sin\frac{n\pi(\mathbf{x}\cdot\mathbf{e}_1)}{l_1}\sin\frac{\pi(\mathbf{x}\cdot\mathbf{e}_2)}{l_2}\sin\frac{\pi(\mathbf{x}\cdot\mathbf{e}_3)}{l_3}, \quad n = 1,2\ldots, \tag{11}$$

$$R\{\delta\mathbf{u}_n\} = \int_{\varkappa} \nabla\otimes\delta\mathbf{u}_n{:}\mathbf{C}{:}\nabla\otimes\delta\mathbf{u}_n\,\mathrm{d}V$$

$$= \frac{\pi^2}{8}\frac{V}{l_1^2}\left(n^2 c + \frac{l_1^2}{l_2^2}\mathbf{e}_2\otimes\mathbf{g}_0:\mathbf{C}:\mathbf{e}_2\otimes\mathbf{g}_0 + \frac{l_1^2}{l_3^2}\mathbf{e}_3\otimes\mathbf{g}_0:\mathbf{C}:\mathbf{e}_3\otimes\mathbf{g}_0\right). \tag{12}$$

From eqn (12) it is clear that $\mathrm{R}\{\delta\mathbf{u}_n\} < 0$ for sufficiently great n. However, the IM in eqn (11) is diffuse (although oscillatory); moreover, for some particular types of materials, non-oscillatory diffuse IMs are also possible.

Hence, for elastic (i.e. incrementally linear) bodies localized instability in the case considered is not indispensable, but for elastic-plastic (i.e. incrementally piecewise-linear) ones the situation is quite different due to inevitable incremental unloading that corresponds to each IM. Indeed, for each non-zero field $\delta\mathbf{u}(\mathbf{x})$ the region $\varkappa$ splits into two subsets: that of incremental loading ($\delta\mathbf{H}:\mathbf{S} > 0$), denoted $\varkappa^{\mathrm{p}}$, and the other of incremental unloading ($\delta\mathbf{H}:\mathbf{S} \leqslant 0$), denoted $\varkappa^{\mathrm{e}}$. Given the boundary conditions in couple with homogeneity, we obtain (by divergence theorem):

$$\int_{\varkappa^{\mathrm{p}}} \delta\mathbf{H}:\mathbf{S}\,\mathrm{d}V + \int_{\varkappa^{\mathrm{e}}} \delta\mathbf{H}:\mathbf{S}\,\mathrm{d}V = \int_{\partial\varkappa} \mathbf{n}\otimes\delta\mathbf{u}:\mathbf{S}\,\mathrm{d}\Sigma = 0. \tag{13}$$

Hence, there are two possibilities: (i) uniformly neutral loading $\delta\mathbf{H}:\mathbf{S} \equiv 0$, for which $\varkappa^{\mathrm{e}} = \varkappa$, the volume V^{e} is equal to the whole volume V and $V^{\mathrm{p}} = 0$ (it will be clear from below that this case gives only positive values to the functional eqn (3); (ii) both sets $\varkappa^{\mathrm{p}}$ and $\varkappa^{\mathrm{e}}$ are not empty and both volumes V^{p} and V^{e} are non-zero whereas their sum is the whole volume V.

Proceeding from such a splitting, we shall find the integral inequality which any primary IM must obey.

4. Inequality for the primary IMs

It is natural to consider all basic elements of the constitutive relation (2.1), namely $\mathbf{C}^{\mathrm{e}}$, $\mathbf{S}$ and θ, as continuously dependent on the loading parameter q. We suppose additionally that the tensor $\mathbf{C}^{\mathrm{e}}(q)$ remains strongly elliptic ($c^{\mathrm{e}}(q) > 0$) for all values of q and that $c^{\mathrm{p}}(q)$ decreases monotonically in q.

Let us show first, that non-zero neutral loading gives only positive values to the functional eqn (3). Indeed, if $\delta\mathbf{H}:\mathbf{S} \equiv 0$, then everywhere in the region $\delta\mathbf{H}:\mathbf{C}^{\mathrm{p}}:\delta\mathbf{H} = \delta\mathbf{H}:\mathbf{C}^{\mathrm{e}}:\delta\mathbf{H}$ and hence,

$$R\{\delta\mathbf{u}\} = \int_{\varkappa} \delta\mathbf{H}:\mathbf{C}^{\mathrm{e}}:\delta\mathbf{H}\,\mathrm{d}V \geqslant c^{\mathrm{e}} \int_{\varkappa} \delta\mathbf{H}:\delta\mathbf{H}\,\mathrm{d}V > 0 \tag{14}$$

(the first inequality follows from Van Hove's theorem and the second, from supposition regarding c^{e}). Thus, IMs by no means correspond to neutral loading, and hence, for any IM the values of V^{p} and V^{e} are both non-zero (Section 3).

It was proved in Section 3, that $R\{\delta\mathbf{u}\} \geqslant 0$ for $c^{\mathrm{p}} \geqslant 0$. Let us show now, that it is actually positive definite even for zero value of c^{p}. There are two distinct cases: when $\delta\mathbf{H}:\mathbf{S} \equiv 0$ (neutral loading) and when it is not. The case of neutral loading has been considered above. In the other case we have:

$$R\{\delta\mathbf{u}\} = \int_{\varkappa} \delta\mathbf{H}{:}\mathbf{C}^{\mathrm{p}}{:}\delta\mathbf{H}\,\mathrm{d}V + \theta\int_{\varkappa^{\mathrm{e}}} (\delta\mathbf{H}{:}\mathbf{S})^2\,\mathrm{d}V \geqslant c^{\mathrm{p}}\int_{\varkappa} (\delta\mathbf{H}{:}\delta\mathbf{H})\,\mathrm{d}V + \theta\int_{\varkappa^{\mathrm{e}}} (\delta\mathbf{H}{:}\mathbf{S})^2\,\mathrm{d}V > 0. \tag{15}$$

Let us remark that with certain type of the moduli tensor, $R\{\delta\mathbf{u}\}$ can be positive definite for $c = 0$ even in elasticity (Hayes, 1966). Summarizing the above and taking the account of the beginning of Section 3, we have

$$c^{\mathrm{p}}(q_*) = 0, \quad c^{\mathrm{p}}(q) > 0 \quad \text{for } q < q_*, \quad c^{\mathrm{p}}(q) < 0 \quad \text{for } q > q_*, \quad c^{\mathrm{p}}(q_*+0) = -0, \tag{16}$$

and $R\{\delta\mathbf{u}\}$ is positive definite for $q \leqslant q_*$, whereas immediately after passing the value $q = q_*$ the functional $R\{\delta\mathbf{u}\}$ loses its positive definiteness, i.e. it can take negative values (the inception of instability, whose earliest instant in a strict sense being absent).

In order to find some basic inequality for the IMs, we introduce for each value of q some auxiliary family of the fourth-order tensors versus a parameter α, that includes $\mathbf{C}^{\mathrm{e}}(q)$ and $\mathbf{C}^{\mathrm{p}}(q)$:

$$\mathbf{C}(q,\alpha) \equiv \mathbf{C}^{\mathrm{e}}(q) - \alpha\mathbf{S}(q)\otimes\mathbf{S}(q), \quad \alpha \geqslant 0, \tag{17}$$

$$\mathbf{C}(q,0) = \mathbf{C}^{\mathrm{e}}(q), \quad \mathbf{C}(q,\theta(q)) = \mathbf{C}^{\mathrm{p}}(q). \tag{18}$$

Consider the corresponding Hadamard number $c(q,\alpha)$ as a function of α. The properties of this function are studied in Appendix 1 and it is found that:

1. The function $c(q,\alpha)$ is continuous and non-increasing in α.
2. When α is varied from zero to $+\infty$, $c(q,\alpha)$ is varied from $c^{\mathrm{e}}(q) > 0$ to $-\infty$.
3. At the values of α, for which $c(q,\alpha) < c^{\mathrm{e}}(q)$, the function $c(q,\alpha)$ is monotonically decreasing in α.

Hence, for any q there is a single value $\alpha_*(q) > 0$ such, that:

$$c(q,\alpha_*(q)) = 0, \tag{19}$$

$$c(q,\alpha) < 0 \Leftrightarrow \alpha > \alpha_*(q), \tag{20}$$

$$c(q,\alpha) = -0 \Leftrightarrow \alpha = \alpha_*(q) + 0. \tag{21}$$

It is not difficult to show that $\alpha_* \geqslant c^{\mathrm{e}}/(\mathbf{S}{:}\mathbf{S})$.

By eqns (16), (18) and (21) we have:

$$c^{\mathrm{p}}(q_*+0) = c(q_*+0,\theta(q_*+0)) = -0 \Rightarrow \theta(q_*+0) = \alpha_*(q_*+0)+0. \tag{22}$$

The value $\mathbf{C}(q,\alpha_*(q))$ we denote $\mathbf{C}_*(q)$. Omitting the argument q, we can write:

$$\mathbf{C}^{\mathrm{e}} = \mathbf{C}_* + \alpha_*\mathbf{S}\otimes\mathbf{S}, \quad \mathbf{C}^{\mathrm{p}} = \mathbf{C}_* - (\theta-\alpha_*)\mathbf{S}\otimes\mathbf{S}. \tag{23}$$

If $q > q_*$, then $\theta - \alpha_* > 0$; for an IM $\delta\mathbf{u}(\mathbf{x})$ the functional takes a negative value:

$$R\{\delta\mathbf{u}\} = \int_{\varkappa} \delta\mathbf{H}{:}\mathbf{C}_*{:}\delta\mathbf{H}\,\mathrm{d}V + \alpha_*\int_{\varkappa^{\mathrm{e}}} (\delta\mathbf{H}{:}\mathbf{S})^2\,\mathrm{d}V - (\theta-\alpha_*)\int_{\varkappa^{p}} (\delta\mathbf{H}{:}\mathbf{S})^2\,\mathrm{d}V < 0. \tag{24}$$

Due to Van Hove's theorem the first integral in eqn (24) is non-negative and hence,

$$\begin{cases} \int_{\varkappa} \delta\mathbf{H}{:}\mathbf{C}_{*}{:}\delta\mathbf{H}\,\mathrm{d}V < (\theta-\alpha_{*})\int_{\varkappa^{\mathrm{p}}} (\delta\mathbf{H}{:}\mathbf{S})^2\,\mathrm{d}V \\ \alpha_{*}\int_{\varkappa^{\mathrm{e}}} (\delta\mathbf{H}{:}\mathbf{S})^2\,\mathrm{d}V < (\theta-\alpha_{*})\int_{\varkappa^{p}} (\delta\mathbf{H}{:}\mathbf{S})^2\,\mathrm{d}V \end{cases} \tag{25}$$

Using the Bunyakovskii–Schwarz inequality for the set $\varkappa^{\mathrm{e}}$ and taking into account eqn (13), we obtain from the second inequality (25):

$$\frac{\theta-\alpha_{*}}{a_{*}} > \frac{1}{V^{\mathrm{e}}}\frac{\left(\int_{\varkappa^{\mathrm{e}}} \delta\mathbf{H}{:}\mathbf{S}\,\mathrm{d}V\right)^2}{\int_{\varkappa^{\mathrm{p}}} (\delta\mathbf{H}{:}\mathbf{S})^2\,\mathrm{d}V} = \frac{1}{4V^{\mathrm{e}}}\frac{\left(\int_{\varkappa} |\delta\mathbf{H}{:}\mathbf{S}|\,\mathrm{d}V\right)^2}{\int_{\varkappa^{\mathrm{p}}} (\delta\mathbf{H}{:}\mathbf{S})^2\,\mathrm{d}V}$$

$$> \frac{1}{4V}\frac{\left(\int_{\varkappa} |\delta\mathbf{H}{:}\mathbf{S}|\,\mathrm{d}V\right)^2}{\int_{\varkappa} (\delta\mathbf{H}{:}\mathbf{S})^2\,\mathrm{d}V} \equiv \frac{1}{4V}V_{\mathrm{loc}}\{\delta\mathbf{H}{:}\mathbf{S}\}. \tag{26}$$

The functional

$$V_{\mathrm{loc}}\{\varphi\} \equiv \frac{\left(\int_{\varkappa} |\varphi(\mathbf{x})|\,\mathrm{d}V\right)^2}{\int_{\varkappa} (\varphi(\mathbf{x}))^2\,\mathrm{d}V} \leqslant V \tag{27}$$

is defined on the set of piecewise continuous functions $\varphi(\mathbf{x})$; we call it the 'localizational volume' corresponding to the field $\varphi(\mathbf{x})$. Its values are never greater than V (due to Bunyakovskii–Schwarz inequality for $\varkappa$) and evaluate the volume of the part of $\varkappa$, where relatively great absolute values of the quantity φ are localized. The quantity of this type was introduced for a one-dimensional continuum by Ryzhak (1993). Detailed consideration of the properties of localizational volume is given in Appendix 2.

For the primary IMs $\theta-\alpha_{*} = +0$. Due to the fact, that $\alpha_{*} > 0$ is always finite, we have:

$$\frac{\theta-\alpha_{*}}{\alpha_{*}} = +0 \quad \Rightarrow \quad \frac{V_{\mathrm{loc}}\{\delta\mathbf{H}{:}\mathbf{S}\}}{V} = +0. \tag{28}$$

The infinitesimal values of localizational volume relative to the whole volume, we treat as a 'volumetric' localization and eqn (28) shows that all the primary IMs are localized in such a volumetric sense.

Note, that the smallness of $V_{\mathrm{loc}}\{\delta\mathbf{H}:\mathbf{S}\}$ does not mean that V^{p} is small. It is not difficult to show that $V_{\mathrm{loc}} \leqslant 4V^{\mathrm{p}}$, and if V^{p} is infinitesimal, then eqn (28) is surely valid and volumetric localization takes place, but V^{p} may be finite as well, while V_{loc} being small and $V^{\mathrm{e}} = V - V^{\mathrm{p}}$ being much greater than V_{loc}.

Actually, it is not difficult to deduce from eqns (25) and (22) a property a bit stronger than eqn (28), precisely the following:

$$\frac{V^{\mathrm{p}}_{\mathrm{loc}}\{\delta \mathbf{H}{:}\mathbf{S}\}}{V^{\mathrm{e}}_{\mathrm{loc}}\{\delta \mathbf{H}{:}\mathbf{S}\}} = +0, \quad \frac{1}{4}V_{\mathrm{loc}} \leqslant V^{\mathrm{p}}_{\mathrm{loc}} \leqslant V^{\mathrm{p}}, \quad V^{\mathrm{e}}_{\mathrm{loc}} \leqslant V^{\mathrm{e}} < V, \tag{29}$$

where it is meant that the integrals in numerator are taken over $\varkappa^{\mathrm{p}}$ and in denominator they are taken over $\varkappa^{\mathrm{e}}$.

5. Some illustrative examples

This section consists of a collection of particular cases of localized instability analysis, that serve to illustrate and clarify different aspects of above general considerations.

5.1. One-dimensional elastic-plastic filament with fixed ends

A merit of one-dimensional model is that due to its simplicity instability analysis can be carried out completely.

Consider a one-dimensional filament stretched to a homogeneous plastic state, after which its ends are fixed. The incremental tension obeys the following elastic-plastic constitutive relation:

$$\delta T_{\varkappa} = \begin{cases} h\delta H, & \delta H > 0 \\ G\delta H, & \delta H \leqslant 0 \end{cases}, \tag{30}$$

$$G > 0, \quad \delta H \equiv \delta u'(x), \quad 0 \leqslant x \leqslant l, \quad \delta u(0) = \delta u(l) = 0, \tag{31}$$

where $\delta u(x)$ is the incremental displacement field vanishing at the ends, and $\delta u'(x)$ is the incremental longitudinal strain.

The functional that governs stability, is a simplified version of eqn (3):

$$R\{\delta u\} = G\int_{\varkappa^{\mathrm{e}}} (\delta u')^2 \, \mathrm{d}x + h\int_{\varkappa^{\mathrm{p}}} (\delta u')^2 \, \mathrm{d}x. \tag{32}$$

The functional $R\{\delta u\}$ is positive definite as long as $h \geqslant 0$, and loses immediately its positive definiteness when h becomes negative (i.e. at $h = -0$).

Suppose that $h < 0$ and $\delta u(x)$ is an IM:

$$R\{\delta u\} < 0 \quad \Leftrightarrow \quad \frac{\int_{\varkappa^{\mathrm{e}}} (\delta u')^2 \, \mathrm{d}x}{\int_{\varkappa^{\mathrm{p}}} (\delta u')^2 \, \mathrm{d}x} < -\frac{h}{G} = \frac{|h|}{G}. \tag{33}$$

Dividing both the numerator and denominator by

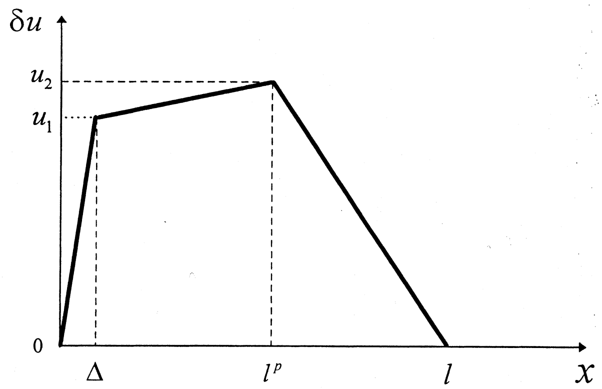

Fig. 1. One-dimensional elastic-plastic filament. A three-interval piecewise-linear displacement field.

$$\left(\int_{\varkappa^{\mathrm{p}}} \delta u' \,\mathrm{d}x\right)^2 = \left(\int_{\varkappa^{\mathrm{e}}} \delta u' \,\mathrm{d}x\right)^2,$$

we find ultimately for the primary instability:

$$\frac{l^{\mathrm{p}}_{\mathrm{loc}}\{\delta u\}}{l^{\mathrm{e}}_{\mathrm{loc}}\{\delta u\}} < \frac{|h|}{G} = +0. \tag{34}$$

Inequality (34) expresses entirely the fact that the IM $\delta u(x)$ is a primary one (i.e. corresponds to $h = -0$): if an IM is primary, then the ratio $l^{\mathrm{p}}_{\mathrm{loc}}/l^{\mathrm{e}}_{\mathrm{loc}}$ is infinitesimal, and conversely, if the ratio is infinitesimal, the corresponding $\delta u(x)$ is a primary IM. Along with it, e.g. l^{p} (unlike $l^{\mathrm{p}}_{\mathrm{loc}}$) may be much greater then $l^{\mathrm{e}} \geqslant l^{\mathrm{e}}_{\mathrm{loc}}$ (where l^{p} and l^{e} are the total lengths of $\varkappa^{\mathrm{p}}$ and $\varkappa^{\mathrm{e}}$ respectively).

Let us show it explicitly for a three-interval piecewise-linear displacement field (Fig. 1):

$$\delta u(x) = \begin{cases} \dfrac{u_1}{\Delta}x, & 0 \leqslant x \leqslant \Delta \\ u_1 + \dfrac{u_2 - u_1}{l^{\mathrm{p}} - \Delta}(x - \Delta), & \Delta \leqslant x \leqslant l^{\mathrm{p}} \\ u_2 - \dfrac{u_2}{l - l^{\mathrm{p}}}(x - l^{\mathrm{p}}), & l^{\mathrm{p}} \leqslant x \leqslant l \end{cases}, \tag{35}$$

$$u_2 > u_1, \quad \varkappa^{\mathrm{p}} = [0, l^{\mathrm{p}}], \quad \varkappa^{\mathrm{e}} = [l^{\mathrm{p}}, l]. \tag{36}$$

Then

$$l^{\mathrm{p}}_{\mathrm{loc}}(\Delta) = \Delta \frac{u_2^2}{u_1^2 + \dfrac{\Delta}{l^{\mathrm{p}} - \Delta}(u_2 - u_1)^2} = \Delta \frac{u_2^2}{u_1^2} + O(\Delta^2)$$

$$l^{\mathrm{p}} = \text{const}, \quad \lim_{\Delta \to 0} l^{\mathrm{p}}_{\mathrm{loc}}(\Delta) = 0$$

$$l^{\mathrm{e}}_{\mathrm{loc}}(\Delta) = l^{\mathrm{e}} = l - l^{\mathrm{p}} = \text{const}, \tag{37}$$

and for any $u_1 < u_2$ and $l^{\mathrm{p}} < l$ there does exist a sufficiently small value of Δ for which the ratio $l^{\mathrm{p}}_{\mathrm{loc}}(\Delta)/l^{\mathrm{e}}_{\mathrm{loc}}$ is also sufficiently small.

5.2. A kind of conventional localization

Firstly, we state the Modified Van Hove's theorem, one of the three modifications of Van Hove's theorem proved by Ryzhak (1993a, 1994), in order to use it in this and the following examples.

5.2.1. Modified Van Hove's theorem

Let the region Ω be a rectangular parallelepiped with the normals to its faces forming the orthonormal triplet $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, let $\mathbf{C}_0 = \text{const}$ be a strongly-elliptic fourth-order tensor (i.e. its Hadamard number $c_0 > 0$), and let $\mathbf{u}(\mathbf{x})$ be a non-zero continuous piecewise-smooth vector field.

If:

(1) the tensor $\mathbf{C}_0$ is specularly symmetric with respect to the plane with normal $\mathbf{e}_1$ and the field $\mathbf{u}(\mathbf{x})$ is tangential on the corresponding pair of faces (i.e. $\mathbf{u} \cdot \mathbf{e}_1 = 0$ there) and vanishes on all the others, then the following inequality is valid:

$$R_0\{\mathbf{u}\} \equiv \int_\Omega \nabla \otimes \mathbf{u} : \mathbf{C}_0 : \nabla \otimes \mathbf{u} \, \mathrm{d}V \geqslant c_0 \int_\Omega \nabla \otimes \mathbf{u} : \nabla \otimes \mathbf{u} \, \mathrm{d}V > 0; \tag{38}$$

(2) the tensor $\mathbf{C}_0$ is orthotropic with the planes of orthotropy parallel to the faces and the field $\mathbf{u}(\mathbf{x})$ is tangential on all the faces (i.e. $\mathbf{u} \cdot \mathbf{e}_i = 0$ on the faces with normals $\pm \mathbf{e}_i$), then inequality (38) is also valid.

Secondly, we note that if a second-order tensor $\mathbf{S}_0$ is specularly symmetric with respect to the plane with normal $\mathbf{e}_i$, then $\mathbf{e}_i \cdot \mathbf{S}_0$ is parallel to $\mathbf{e}_i$. Hence,

$$\int_\Omega \nabla \otimes \mathbf{u} : \mathbf{S}_0 \, \mathrm{d}V = \int_{\partial\Omega} \mathbf{n} \otimes \mathbf{u} : \mathbf{S}_0 \, \mathrm{d}\Sigma = \int_{\partial\Omega} \mathbf{n} \cdot \mathbf{S}_0 \cdot \mathbf{u} \, \mathrm{d}\Sigma = 0 \tag{39}$$

under the hypotheses of both parts of the Modified Van Hove's theorem, provided that $\mathbf{S}_0$ is supposed to possess the same type of symmetry as $\mathbf{C}_0$. Equality (39) conicides with eqn (13).

Thus, all the results of Section 4 remain valid for a homogeneous elastic-plastic body not only under zero displacement boundary conditions, but also under boundary conditions of sliding on a pair of parallelepiped's faces or on all its faces, provided that corresponding material symmetry takes place.

Now we pass to the second example itself. Let the elastic-plastic body be characterized by the moduli tensors

$$\mathbf{C}^{\mathrm{e}} = 2G(\mathbf{1}^{\mathrm{def}} - \tfrac{1}{3}\mathbf{I} \otimes \mathbf{I}) + K\mathbf{I} \otimes \mathbf{I}$$

$$\mathbf{C}^{\mathrm{p}}(h) = 2G(\mathbf{1}^{\mathrm{def}} - \tfrac{1}{3}\mathbf{I} \otimes \mathbf{I} - \mathbf{S} \otimes \mathbf{S}) + K\mathbf{I} \otimes \mathbf{I} + 2h\mathbf{S} \otimes \mathbf{S}, \tag{40}$$

where $\mathbf{I}$ is the second-order unit tensor, $\mathbf{1}^{\text{def}}$ is the fourth-order tensor that maps any second-order tensor $\mathbf{H}$ into its symmetric part:

$$\mathbf{1}^{\text{def}}:\mathbf{H} = \tfrac{1}{2}(\mathbf{H}+\mathbf{H}^T), \tag{41}$$

and $\mathbf{S}$ is symmetric normalized second-order deviator:

$$\mathbf{S} = \mathbf{S}^T, \quad \mathbf{S}:\mathbf{S} = 1, \quad \mathbf{S}:\mathbf{I} = 0. \tag{42}$$

It is clear from eqn (40), that the body is Hookian (with shear modulus G and bulk modulus K) under elastic unloading, and its plastic response is of von Mises type (with plastic shear modulus h). The body is supposed to occupy the region $\varkappa$ of the shape of rectangular parallelepiped

$$0 \leqslant \mathbf{x}\cdot\mathbf{e}_i \leqslant l_i, \quad \mathbf{e}_i\cdot\mathbf{e}_j = \delta_{ij}, \quad i,j = 1,2,3 \tag{43}$$

with zero displacement boundary conditions (clamping) on the faces with normals $\pm\mathbf{e}_2$, $\pm\mathbf{e}_3$ (the second and the third pairs of faces) and sliding boundary conditions on the faces with normals $\pm\mathbf{e}_1$ (the first pair of faces). The tensor $\mathbf{S}$ is specified by the equality

$$\mathbf{S} = \frac{1}{\sqrt{2}}(\mathbf{e}_2\otimes\mathbf{e}_3+\mathbf{e}_3\otimes\mathbf{e}_2). \tag{44}$$

In that case the tensor $\mathbf{C}^{\text{e}}$ is isotropic and the $\mathbf{C}^{\text{p}}$ and $\mathbf{S}$ tensors are specularly symmetric with respect to the plane of the first pair of faces. Thus, the assumptions of the Modified Van Hove's theorem (part 1) are fulfilled together with eqn (39), and hence all the preceeding theory (Section 4) is valid for the body under consideration.

It is not difficult to show that

$$c^{\text{p}}(h) > 0, \quad h > 0; \quad c^{\text{p}}(0) = 0, \quad \mathbf{C}_* = 2G(\mathbf{1}^{\text{def}} - \tfrac{1}{3}\mathbf{I}\otimes\mathbf{I} - \mathbf{S}\otimes\mathbf{S}) + K\mathbf{I}\otimes\mathbf{I} \tag{45}$$

$$\mathbf{e}_3\otimes\mathbf{e}_2:\mathbf{C}_*:\mathbf{e}_3\otimes\mathbf{e}_2 = \mathbf{e}_2\otimes\mathbf{e}_3:\mathbf{C}_*:\mathbf{e}_2\otimes\mathbf{e}_3 = 0; \quad \mathbf{e}_2\otimes\mathbf{e}_3:\mathbf{C}^{\text{p}}(h):\mathbf{e}_2\otimes\mathbf{e}_3 < 0, \quad h < 0.$$

We set the incremental displacement field satisfying the boundary conditions (Fig. 2), by the equality

$$\delta\mathbf{u}(\mathbf{x}) = \begin{cases} \delta\gamma\mathbf{e}_2\left(\sin\pi\dfrac{x_2}{l_2}\right)x_3, & 0 \leqslant x_3 \leqslant \Delta \\ \delta\gamma\mathbf{e}_2\left(\sin\pi\dfrac{x_2}{l_2}\right)\dfrac{\Delta}{l_3-\Delta}(l_3-x_3), & \Delta \leqslant x_3 \leqslant l_3 \end{cases}. \tag{46}$$

Then, given a negative h (with extremely small absolute value), we are always able to choose a sufficiently small value of Δ, so that the field $\delta\mathbf{u}(\mathbf{x})$ be a corresponding IM (a kind of conventional primary localized IM).

It is clear that the $\varkappa^{\text{p}}$ subregion corresponds to the interval $0 \leqslant x_3 \leqslant \Delta$, and the $\varkappa^{\text{e}}$ corresponds to that of $\Delta \leqslant x_3 \leqslant l_3$. For the localizational volumes calculations yield:

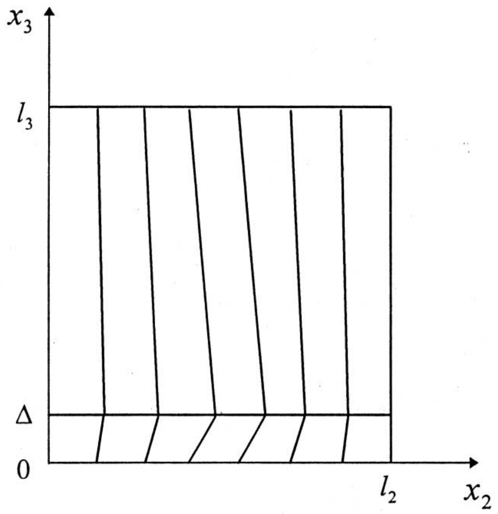

Fig. 2. A kind of conventional localization pattern: the shear band.

$$V^{\mathrm{p}}_{\mathrm{loc}} \equiv \frac{\left(\int_{\varkappa^{\mathrm{p}}} \delta \mathbf{H}{:}\mathbf{S}\,\mathrm{d}V\right)^2}{\int_{\varkappa^{\mathrm{p}}} (\delta \mathbf{H}{:}\mathbf{S})^2\,\mathrm{d}V} = \frac{2}{\pi} l_1 l_2 \Delta = \frac{2}{\pi} V^{\mathrm{p}},$$

$$V^{\mathrm{e}}_{\mathrm{loc}} = \frac{2}{\pi} l_1 l_2 (l_3 - \Delta) = \frac{2}{\pi} V^{\mathrm{e}},$$

$$V_{\mathrm{loc}} = \frac{8}{\pi} \frac{\Delta}{l_3} \frac{l_3 - \Delta}{l_3} V,$$

$$\Delta = +0 \Rightarrow \begin{cases} V_{\mathrm{loc}}/V = +0 \\ V^{\mathrm{p}}_{\mathrm{loc}}/V^{\mathrm{e}}_{\mathrm{loc}} = +0 \end{cases}. \tag{47}$$

From eqn (47) it is clear that conventional primary localized IM eqn (46) is in fact characterized by infinitesimal values of the ratios V_{loc}/V and $V^{\mathrm{p}}_{\mathrm{loc}}/V^{\mathrm{e}}_{\mathrm{loc}}$.

5.3. 'Uniaxial' plasticity and quasi-one-dimensional (quasi-conventional) localization

Let us introduce some 'unusual' incremental elastic-plastic constitutive relation:

$$\mathbf{C}^{\mathrm{e}} = 2G\mathbf{1}^{\mathrm{def}}, \quad \mathbf{C}^{\mathrm{p}}(h) = 2G(\mathbf{1}^{\mathrm{def}} - \mathbf{S} \otimes \mathbf{S}) + 2h\mathbf{S} \otimes \mathbf{S}, \quad \mathbf{S} = \mathbf{e}_3 \otimes \mathbf{e}_3. \tag{48}$$

The only reason for calling it 'unusual' is the choice of $\mathbf{S}$ (the normal to yield surface in the strain space); in all other aspects it is quite a conventional elastoplasticity.

The region $\varkappa$ is again a rectangular parallelepiped, eqn (43), but under different boundary conditions: sliding is supposed on all the faces.

All the suppositions correspond to the second part of the Modified Van Hove's theorem and

$$c^{\mathrm{p}}(h) = \mathbf{e}_3 \otimes \mathbf{e}_3 : \mathbf{C}^{\mathrm{p}}(h) : \mathbf{e}_3 \otimes \mathbf{e}_3 = 2h \begin{cases} >0, & h>0 \\ =0, & h=0 \\ <0, & h<0, \end{cases}$$

$$\mathbf{C}_* = \mathbf{C}^{\mathrm{p}}(0) = 2G(\mathbf{1}^{\mathrm{def}} - \mathbf{S} \otimes \mathbf{S}). \tag{49}$$

Thus, the body is stable as long as $h \geqslant 0$ and the onset of instability is related to the value $h = -0$.

Let us analyse the inequality for IMs, taking the account of simple special structure of the constitutive relation (48). Representing $\delta\mathbf{u}(\mathbf{x})$ and its gradient as

$$\delta\mathbf{u}(\mathbf{x}) = \mathbf{e}_1 \delta u_1(\mathbf{x}) + \mathbf{e}_2 \delta u_2(\mathbf{x}) + \mathbf{e}_3 \delta u_3(\mathbf{x}) \equiv \delta\mathbf{u}_\perp(\mathbf{x}) + \mathbf{e}_3 \delta u_3(\mathbf{x}), \tag{50}$$

$$\begin{aligned} \nabla \otimes \delta\mathbf{u} &= \mathbf{e}_1 \otimes \frac{\partial(\delta\mathbf{u})}{\partial x_1} + \mathbf{e}_2 \otimes \frac{\partial(\delta\mathbf{u})}{\partial x_2} + \mathbf{e}_3 \otimes \frac{\partial(\delta\mathbf{u})}{\partial x_3} = \nabla_\perp \otimes \delta\mathbf{u} + \mathbf{e}_3 \otimes \frac{\partial(\delta\mathbf{u})}{\partial x_3} \\ &= \nabla_\perp \otimes \delta\mathbf{u}_\perp + \left(\mathbf{e}_3 \otimes \frac{\partial(\delta\mathbf{u}_\perp)}{\partial x_3} + \nabla_\perp(\delta u_3) \otimes \mathbf{e}_3\right) + \mathbf{e}_3 \otimes \mathbf{e}_3 \frac{\partial(\delta u_3)}{\partial x_3} \end{aligned} \tag{51}$$

we get:

$$\begin{aligned} \nabla \otimes \delta\mathbf{u} : \mathbf{1}^{\mathrm{def}} : \nabla \otimes \delta\mathbf{u} &= \nabla_\perp \otimes \delta\mathbf{u}_\perp : \mathbf{1}^{\mathrm{def}} : \nabla_\perp \otimes \delta\mathbf{u}_\perp \\ &+ \left(\mathbf{e}_3 \otimes \frac{\partial(\delta\mathbf{u}_\perp)}{\partial x_3} + \nabla_\perp(\delta u_3) \otimes \mathbf{e}_3\right) : \mathbf{1}^{\mathrm{def}} : \left(\mathbf{e}_3 \otimes \frac{\partial(\delta\mathbf{u}_\perp)}{\partial x_3} + \nabla_\perp(\delta u_3) \otimes \mathbf{e}_3\right) + \left(\frac{\partial(\delta u_3)}{\partial x_3}\right)^2, \end{aligned}$$

$$\nabla \otimes \delta\mathbf{u} : \mathbf{S} = \frac{\partial(\delta u_3)}{\partial x_3},$$

$$\begin{aligned} 0 > R\{\delta\mathbf{u}\} &= 2G \int_\varkappa \nabla_\perp \otimes \delta\mathbf{u}_\perp : \mathbf{1}^{\mathrm{def}} : \nabla_\perp \otimes \delta\mathbf{u}_\perp \, \mathrm{d}V \\ &+ 2G \int_\varkappa \left(\mathbf{e}_3 \otimes \frac{\partial(\delta\mathbf{u}_\perp)}{\partial x_3} + \nabla_\perp(\delta u_3) \otimes \mathbf{e}_3\right) : \mathbf{1}^{\mathrm{def}} : \left(\mathbf{e}_3 \otimes \frac{\partial(\delta\mathbf{u}_\perp)}{\partial x_3} + \nabla_\perp(\delta u_3) \otimes \mathbf{e}_3\right) \mathrm{d}V \\ &+ 2G \int_{\varkappa^{\mathrm{e}}} \left(\frac{\partial \delta u_3}{\partial x_3}\right)^2 \mathrm{d}V + 2h \int_{\varkappa^{\mathrm{p}}} \left(\frac{\partial \delta u_3}{\partial x_3}\right)^2 \mathrm{d}V \\ &\geqslant G \int_\varkappa \nabla_\perp \otimes \delta\mathbf{u}_\perp : \nabla_\perp \otimes \delta\mathbf{u}_\perp \, \mathrm{d}V + 2G \int_{\varkappa^{\mathrm{e}}} \left(\frac{\partial \delta u_3}{\partial x_3}\right)^2 \mathrm{d}V + 2h \int_{\varkappa^{\mathrm{p}}} \left(\frac{\partial \delta u_3}{\partial x_3}\right)^2 \mathrm{d}V \\ &\Rightarrow \frac{1}{2} \int_\varkappa \nabla_\perp \otimes \delta\mathbf{u}_\perp : \nabla_\perp \otimes \delta\mathbf{u}_\perp \, \mathrm{d}V + \int_{\varkappa^{\mathrm{e}}} \left(\frac{\partial \delta u_3}{\partial x_3}\right)^2 \mathrm{d}V < -\frac{h}{G} \int_{\varkappa^{\mathrm{p}}} \left(\frac{\partial \delta u_3}{\partial x_3}\right)^2 \mathrm{d}V, \quad h = -0, \end{aligned}$$

$$\Rightarrow \begin{cases} \dfrac{\displaystyle\int_{\varkappa} \nabla_\perp \otimes \delta \mathbf{u}_\perp : \nabla_\perp \otimes \delta \mathbf{u}_\perp \,\mathrm{d}V}{\displaystyle\int_{\varkappa} \left(\frac{\partial \delta u_3}{\partial x_3}\right)^2 \mathrm{d}V} < \dfrac{|h|}{G} = +0 \\ \dfrac{\displaystyle\int_{\varkappa^{\mathrm{e}}} \left(\frac{\partial \delta u_3}{\partial x_3}\right)^2 \mathrm{d}V}{\displaystyle\int_{\varkappa^{\mathrm{p}}} \left(\frac{\partial \delta u_3}{\partial x_3}\right)^2 \mathrm{d}V} < \dfrac{|h|}{G} = +0 \end{cases}. \tag{52}$$

Firstly, we note that $\delta \mathbf{u}_\perp$ vanishes on side edges (parallel to $\mathbf{e}_3$) of the parallelepiped, and relative smallness of its gradient in the plane perpendicular to $\mathbf{e}_3$, results in its relative smallness as compared to δu_3. Secondly, noting that

$$\int_0^{l_3} \frac{\partial \delta u_3}{\partial x_3} \mathrm{d}x_3 = 0 = \int_{l_3^{\mathrm{e}}(\mathbf{x}_\perp)} \frac{\partial \delta u_3}{\partial x_3} \mathrm{d}x_3 + \int_{l_3^{\mathrm{p}}(\mathbf{x}_\perp)} \frac{\partial \delta u_3}{\partial x_3} \mathrm{d}x_3, \quad \forall \mathbf{x}_\perp,$$

and using the Bunyakovskii–Schwarz inequality for $l_3^{\mathrm{e}}(\mathbf{x}_\perp)$:

$$\int_{l_3^{\mathrm{e}}(\mathbf{x}_\perp)} \left(\frac{\partial \delta u_3}{\partial x_3}\right)^2 \mathrm{d}x_3 \geqslant \frac{1}{4l_3} \left(\int_0^{l_3} \left|\frac{\partial \delta u_3}{\partial x_3}\right| \mathrm{d}x_3\right)^2,$$

we get:

$$\frac{1}{4l_3} \frac{\displaystyle\int_0^{l_1}\int_0^{l_2} \mathrm{d}x_1\,\mathrm{d}x_2 \left(\int_0^{l_3} \left|\frac{\partial \delta u_3}{\partial x_3}\right| \mathrm{d}x_3\right)^2}{\displaystyle\int_0^{l_1}\int_0^{l_2} \mathrm{d}x_1\,\mathrm{d}x_2 \int_0^{l_3} \left(\frac{\partial \delta u_3}{\partial x_3}\right)^2 \mathrm{d}x_3} = +0.$$

Denoting by $\alpha(\mathbf{x}_\perp) = \alpha(x_1, x_2)$ the ratio

$$\alpha(\mathbf{x}_\perp) \equiv \frac{\displaystyle\int_0^{l_3} \left(\frac{\partial \delta u_3}{\partial x_3}\right)^2 \mathrm{d}x_3}{\displaystyle\int_0^{l_1}\int_0^{l_2} \mathrm{d}x_1\,\mathrm{d}x_2 \int_0^{l_3} \left(\frac{\partial \delta u_3}{\partial x_3}\right)^2 \mathrm{d}x_3}, \quad \int_0^{l_1}\int_0^{l_2} \alpha(\mathbf{x}_\perp)\,\mathrm{d}x_1\,\mathrm{d}x_2 = 1,$$

we obtain ultimately:

$$\frac{1}{4l_3}\int_0^{l_1}\int_0^{l_2} \alpha(\mathbf{x}_\perp) l_{3\mathrm{loc}}(\mathbf{x}_\perp)\,\mathrm{d}x_1\,\mathrm{d}x_2 = \frac{\langle l_{3\mathrm{loc}}\rangle^{(\alpha)}}{4l_3} = +0, \quad l_{3\mathrm{loc}}(\mathbf{x}_\perp) \equiv \frac{\left(\displaystyle\int_0^{l_3} \left|\frac{\partial \delta u_3}{\partial x_3}\right| \mathrm{d}x_3\right)^2}{\displaystyle\int_0^{l_3} \left(\frac{\partial \delta u_3}{\partial x_3}\right)^2 \mathrm{d}x_3}. \tag{53}$$

Here $l_{3\mathrm{loc}}\,(\mathbf{x}_\perp)$ is the 'localizational length', that characterizes localization in the $\mathbf{e}_3$ direction, and $\langle l_{3\mathrm{loc}}\rangle^{(\alpha)}$ is a result of its averaging (with the weight $\alpha(\mathbf{x}_\perp)$) over the rectangle $0 \leqslant x_1 \leqslant l_1, 0 \leqslant x_2 \leqslant l_2$.

Thus, from eqns (52), (53) it is clear that all the primary IMs in the case considered are characterized by quasi-one-dimensional localization (in the $\mathbf{e}_3$ direction). It is not a conventional localization within a thin layer, but nevertheless, some kind of localized stratification can be observed here.

5.4. Purely volumetric plasticity and a possibility in principle of completely non-conventional localization

This example serves as an illustration of the fact that the mode of localization can be completely different from localization in a layer or from any kind of stratification similar to that of the previous example. For some hypothetical volumetric plasticity (unusual, but of quite a conventional structure) a localized IM is found (a possible one, but not the only possible) with a small sphere as a localization zone. Since a sphere has no other geometrical parameters but radius, which is equivalent to volume, we come to the conclusion that at least for this very type of plasticity the infinitesimal localizational volume is the only feature characterizing the whole class of primary IMs.

We specify incremental elastic-plastic constitutive relation by the equalities:

$$\mathbf{C}^{\mathrm{e}} = 2G(\mathbf{1}^{\mathrm{def}} - \tfrac{1}{3}\mathbf{I}\otimes\mathbf{I}) + K\mathbf{I}\otimes\mathbf{I}, \quad G>0, \quad K>0,$$

$$\mathbf{C}^{\mathrm{p}}(h) = 2G(\mathbf{1}^{\mathrm{def}} - \tfrac{1}{3}\mathbf{I}\otimes\mathbf{I}) + h\mathbf{I}\otimes\mathbf{I}, \quad \mathbf{S} = \frac{1}{\sqrt{3}}\mathbf{I}. \tag{54}$$

For both branches it has the form of Hooke's law with different bulk moduli.

It is not difficult to show that

$$c^{\mathrm{p}}(h) = \min(G, h + \tfrac{4}{3}G),$$

$$c^{\mathrm{p}}(h_*) = 0 \Rightarrow h_* = -\tfrac{4}{3}G, \quad \mathbf{C}_* = \mathbf{C}(h_*) = 2G(\mathbf{1}^{\mathrm{def}} - \mathbf{I}\otimes\mathbf{I}). \tag{55}$$

The constitutive relation (54) is isotropic, and hence, it may equally be assumed, that a body has arbitrary shape with zero displacement boundary conditions, or that it has a special shape and boundary conditions corresponding to any part of the Modified Van Hove's theorem. We choose the suppositions corresponding to the second part of the theorem: parallelepiped as a region and sliding all over the surface.

Representation (24) takes the form:

$$R\{\delta\mathbf{u}\} = 2G\int_{\varkappa} \nabla\otimes\delta\mathbf{u}:(\mathbf{1}^{\mathrm{def}} - \mathbf{I}\otimes\mathbf{I}):\nabla\otimes\delta\mathbf{u}\,\mathrm{d}V$$

$$+ \left(K + \frac{4}{3}G\right)\int_{\varkappa^{\mathrm{e}}} (\nabla\cdot\delta\mathbf{u})^2\,\mathrm{d}V + \left(h + \frac{4}{3}G\right)\int_{\varkappa^{\mathrm{p}}} (\nabla\cdot\delta\mathbf{u})^2\,\mathrm{d}V < 0.$$

Using the well-known Kelvin formula, the first integral can be converted into

$$G\int_{\varkappa} |\nabla\times\delta\mathbf{u}|^2\,\mathrm{d}V$$

and we get:

$$G\int_{\varkappa} |\nabla\times\delta\mathbf{u}|^2\,\mathrm{d}V + \left(K+\frac{4}{3}G\right)\int_{\varkappa^{\mathrm{e}}} (\nabla\cdot\delta\mathbf{u})^2\,\mathrm{d}V < \left((-h)-\frac{4}{3}G\right)\int_{\varkappa^{\mathrm{p}}} (\nabla\cdot\delta\mathbf{u})^2\,\mathrm{d}V,$$

$$(-h)-\frac{4}{3}G = +0 \quad (56)$$

$$\Leftrightarrow \frac{\displaystyle\int_{\varkappa} |\nabla\times\delta\mathbf{u}|^2\,\mathrm{d}V}{\displaystyle\int_{\varkappa^{\mathrm{p}}} (\nabla\cdot\delta\mathbf{u})^2\,\mathrm{d}V} = +0, \quad \frac{\displaystyle\int_{\varkappa^{\mathrm{e}}} (\nabla\cdot\delta\mathbf{u})^2\,\mathrm{d}V}{\displaystyle\int_{\varkappa^{\mathrm{p}}} (\nabla\cdot\delta\mathbf{u})^2\,\mathrm{d}V} = +0. \quad (57)$$

Trying potential fields $\delta\mathbf{u}(\mathbf{x})$, for which $\nabla\times\delta\mathbf{u}\equiv 0$, we need only to fulfill the second relation (57). Let us make use of some potential fields known from electrostatics; precisely, we take as $\delta\mathbf{u}(\mathbf{x})$ the electric field of a pair of homogeneous distributions of charge, the first being a sphere of small radius and the second being a concentric spherical layer of relatively large external radius (Fig. 3), whose total charge is opposite to that of a sphere. The field vanishes outside the layer and the

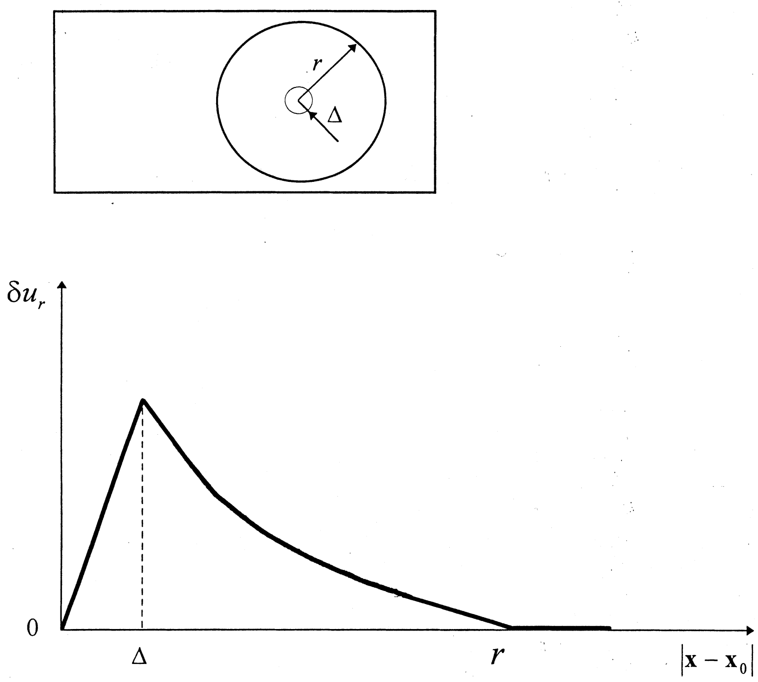

Fig. 3. A kind of completely non-conventional localization pattern: the spherical zone of localized expansion.

boundary conditions are fulfilled provided that the layer lies within the body. Divergence of the field is proportional to the density of charge. If Δ is internal radius, r is external one, and $q > 0$ is a total charge of a sphere of radius Δ, then:

$$\nabla \cdot \delta \mathbf{u} = 4\pi q / \tfrac{4}{3}\pi\Delta^3 > 0, \quad |\mathbf{x} - \mathbf{x}_0| \leqslant \Delta,$$

$$\nabla \cdot \delta \mathbf{u} = -4\pi q / \tfrac{4}{3}\pi (r^3 - \Delta^3) \leqslant 0, \quad \Delta < |\mathbf{x} - \mathbf{x}_0| \leqslant r,$$

$$V^{\mathrm{p}}_{\mathrm{loc}} = V^{\mathrm{p}} = \tfrac{4}{3}\pi\Delta^3, \quad V^{\mathrm{e}}_{\mathrm{loc}} = \tfrac{4}{3}\pi(r^3 - \Delta^3) \tag{58}$$

$$\frac{V^{\mathrm{p}}_{\mathrm{loc}}}{V^{\mathrm{e}}_{\mathrm{loc}}} = \frac{\displaystyle\int_{\varkappa^{\mathrm{e}}} (\nabla \cdot \delta \mathbf{u})^2 \, \mathrm{d}V}{\displaystyle\int_{\varkappa^{\mathrm{p}}} (\nabla \cdot \delta \mathbf{u})^2 \, \mathrm{d}V} = \frac{\Delta^3}{r^3 - \Delta^3} = +0 \Leftrightarrow \frac{\Delta}{r} = +0. \tag{59}$$

Some different IMs of the same type, but with localization within a thin cylinder can be obtained using the fields of homogeneously charged concentric cylinders, provided that their common axis is normal to some pair of faces. This cylindrical type of localization is possible not only for a 'volumetric' plasticity, eqn (54), but also for a family of elastic-plastic relations with 'cylindrical' plasticity:

$$\mathbf{C}^{\mathrm{e}} = 2G\mathbf{1}^{\mathrm{def}}, \quad \mathbf{C}^{\mathrm{p}} = 2G(\mathbf{1}^{\mathrm{def}} - \mathbf{S} \otimes \mathbf{S}) + 2h\mathbf{S} \otimes \mathbf{S},$$

$$\mathbf{S} = \frac{\cos\alpha}{\sqrt{2}}(\mathbf{I} - \mathbf{e}_i \otimes \mathbf{e}_i) + \sin\alpha \mathbf{e}_i \otimes \mathbf{e}_i, \quad 0 \leqslant \alpha < \arcsin\left(\frac{1}{\sqrt{3}}\right). \tag{60}$$

6. Concluding remarks

It is proved that localization of incremental deformation (understood in some generalized sense) is an attribute of primary instability in homogeneous elastic-plastic bodies with constrained boundary. The analysis is free from any usual a priori suppositions regarding the IMs (e.g. from that of shear banding). As mentioned in Section 1, indispensability of localization is a global non-linear effect, the one being absent in incrementally linear solids even under suitable boundary conditions (i.e. diffuse IMs are possible there as well as localized ones).

Homogeneity of a body in couple with rigid boundary constraint are the typical conditions for localization to be possible in principle (i.e. for stability to be preserved up to violation of Hadamard's inequality). However, there exist some different conditions, that admit localization: e.g. the body may be non-homogeneous (Nikitin and Ryzhak, 1986) or the boundary constraint may be not absolutely rigid (Ryzhak, 1993, 1993a, b, 1994). Although the analysis for those cases (except for some of the examples in Section 5) is not presented in the paper, we state that it leads to mainly the same result, by means of the modified method of Ryzhak (1993).

It is clear, that found here, the smallness of localizational volume does not exhaust all the properties of the primary IMs in some specific cases, since we take into account only non-negativeness of the first integral in eqn (24), which is in fact infinitesimal example 5.3 in Section 5 illustrates such a conclusion. Nevertheless, example 5.4 in Section 5 (with purely volumetric plasticity) shows, that a spherical zone of localization is possible in principle, and hence, for that case the smallness of localizational volume is the only geometrical property characterizing the whole class of primary IMs: a sphere has no other parameters but volume. Thus, in the most general setting of the problem of localized primary instability, the smallness of localizational volume is maximal geometrical characterization of that type of instability.

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Appendix A1: Properties of the function $c(\alpha)$

In examining the properties of $c(q, \alpha)$ as a function of α, we omit the argument q:

$$\mathbf{C}(\alpha) \equiv \mathbf{C}^{\mathrm{e}} - \alpha \mathbf{S} \otimes \mathbf{S}, \tag{A1.1}$$

$$c(\alpha) \equiv \min_{|\mathbf{f}| = |\mathbf{g}| = 1} \mathbf{f} \otimes \mathbf{g}{:}\mathbf{C}(\alpha){:}\mathbf{f} \otimes \mathbf{g} = \mathbf{f}_0(\alpha) \otimes \mathbf{g}_0(\alpha){:}\mathbf{C}(\alpha){:}\mathbf{f}_0(\alpha) \otimes \mathbf{g}_0(\alpha) \tag{A1.2}$$

(continuity, closedness and boundedness $\Rightarrow$ minimum is attained). By assumption,

$$c^{\mathrm{e}} = c(0) > 0. \tag{A1.3}$$

The second-order tensor $\mathbf{S}$ in non-zero, hence, there exists a pair of unit vectors $\mathbf{f}_1$, $\mathbf{g}_1$ such, that

$$\mathbf{f}_1 \otimes \mathbf{g}_1{:}\mathbf{S} \neq 0, \tag{A1.4}$$

$$c(\alpha) \leqslant \mathbf{f}_1 \otimes \mathbf{g}_1{:}\mathbf{C}^{\mathrm{e}}{:}\mathbf{f}_1 \otimes \mathbf{g}_1 - \alpha(\mathbf{f}_1 \otimes \mathbf{g}_1{:}\mathbf{S})^2 \Rightarrow c(\alpha) \to -\infty \quad \text{for } \alpha \to +\infty, \tag{A1.5}$$

i.e. $c(\alpha)$ varies from $c^{\mathrm{e}} > 0$ to $-\infty$. Let β be positive. Then

$$c(\alpha+\beta) \leqslant \mathbf{f}_0(\alpha) \otimes \mathbf{g}_0(\alpha){:}\mathbf{C}(\alpha+\beta){:}\mathbf{f}_0(\alpha) \otimes \mathbf{g}_0(\alpha) = c(\alpha) - \beta(\mathbf{f}_0(\alpha) \otimes \mathbf{g}_0(\alpha){:}\mathbf{S})^2 \leqslant c(\alpha), \tag{A1.6}$$

which proves that $c(\alpha)$ is non-increasing in α. Moreover, if $c(\alpha) < c^{\mathrm{e}} = c(0)$, then we have:

$$\begin{aligned} c^{\mathrm{e}} > c(\alpha) &= \mathbf{f}_0(\alpha) \otimes \mathbf{g}_0(\alpha){:}\mathbf{C}^{\mathrm{e}}{:}\mathbf{f}_0(\alpha) \otimes \mathbf{g}_0(\alpha) - \alpha(\mathbf{f}_0(\alpha) \otimes \mathbf{g}_0(\alpha){:}\mathbf{S})^2 \\ &\geqslant c^{\mathrm{e}} - \alpha(\mathbf{f}_0(\alpha) \otimes \mathbf{g}_0(\alpha){:}\mathbf{S})^2 \\ &\Rightarrow \alpha(\mathbf{f}_0(\alpha) \otimes \mathbf{g}_0(\alpha){:}\mathbf{S})^2 > 0 \\ &\Rightarrow \alpha > 0 \quad \text{and} \quad (\mathbf{f}_0(\alpha) \otimes \mathbf{g}_0(\alpha){:}\mathbf{S})^2 > 0. \end{aligned} \tag{A1.7}$$

Hence, if $c(\alpha) < c^{\mathrm{e}}$, then by eqns (A1.6) and (A1.7) for $\beta > 0$ we obtain the strict inequality:

$$c(\alpha+\beta) < c(\alpha) \tag{A1.8}$$

i.e. for such values of α the function $c(\alpha)$ is monotonically decreasing.

To prove continuity we note that for $\beta > 0$

$$c(\alpha+\beta) = \min_{|\mathbf{f}|=|\mathbf{g}|=1} \mathbf{f}\otimes\mathbf{g}(\mathbf{C}(\alpha)-\beta\mathbf{S}\otimes\mathbf{S}){:}\mathbf{f}\otimes\mathbf{g}$$

$$\geqslant \min_{|\mathbf{f}|=|\mathbf{g}|=1} \mathbf{f}\otimes\mathbf{g}{:}\mathbf{C}(\alpha){:}\mathbf{f}\otimes\mathbf{g} - \beta \max_{|\mathbf{f}|=|\mathbf{g}|=1} (\mathbf{f}\otimes\mathbf{g}{:}\mathbf{S})^2 \geqslant c(\alpha)-\beta\mathbf{S}{:}\mathbf{S}, \tag{A1.9}$$

which, together with eqn (A1.6) means continuity: $|c(\alpha_1)-c(\alpha_2)| \leqslant |\alpha_1-\alpha_2|\mathbf{S}{:}\mathbf{S}$.

Thus, all the properties mentioned and used in Section 4, are proved together with relations (19)–(21), that result from them.

Appendix A2: Properties of the functional $V_{\mathrm{loc}}\{\varphi\}$

Here we establish some properties of the functional $V_{\mathrm{loc}}\{\varphi\}$ which enable us to consider it as a measure of localization inherent to some scalar field $\varphi(\mathbf{x})$ (in the paper the meaning of this field is some characteristic incremental strain $\varphi = \delta\mathbf{H}:\mathbf{S} = \frac{1}{2}(\delta\mathbf{H}+\delta\mathbf{H}^T){:}\mathbf{S})$.

Because of supposed piecewise smoothness of the incremental displacements $\delta\mathbf{u}(\mathbf{x})$, it is natural to consider $\varphi(\mathbf{x})$ as a piecewise-continuous scalar field with the set of positiveness $\varkappa^{\mathrm{p}}$, the latter being a unity of a number of bounded regions:

$$\varphi(\mathbf{x}) > 0, \quad \mathbf{x}\in\varkappa^{\mathrm{p}} \tag{A2.1}$$

The Bunyakovskii–Schwarz inequality for $\varkappa$ gives:

$$\left(\int_{\varkappa} |\varphi|\,\mathrm{d}V\right)^2 \leqslant V\int_{\varkappa} \varphi^2\,\mathrm{d}V, \tag{A2.2}$$

from which we obtain immediately:

$$V_{\mathrm{loc}}\{\varphi\} \equiv \frac{\left(\int_{\varkappa} |\varphi|\,\mathrm{d}V\right)^2}{\int_{\varkappa} \varphi^2\,\mathrm{d}V} = \frac{\langle|\varphi|\rangle^2}{\langle\varphi^2\rangle} V \leqslant V, \quad \frac{V_{\mathrm{loc}}\{\varphi\}}{V} = \frac{\langle|\varphi|\rangle^2}{\langle\varphi^2\rangle} \leqslant 1, \tag{A2.3}$$

the equality holding if and only if $|\varphi(\mathbf{x})|$ is constant over $\varkappa$. Note that $V_{\mathrm{loc}}\{\varphi\}$ is invariant with respect to multiplying $\varphi(\mathbf{x})$ by any non-zero constant.

Let us consider first, two types of transformation of a field $\varphi(\mathbf{x})$, that we call its 'localization' and 'homogenization'.

A2.1. Localization

We isolate some subset $\varkappa_0$ in $\varkappa$, with the volume $V_0 < V$, and transform $\varphi(\mathbf{x})$ into $\varphi_a(\mathbf{x})$ as follows:

$$\varphi_a(\mathbf{x}) \equiv \begin{cases} \varphi(\mathbf{x}) + \alpha, & \mathbf{x} \in \varkappa_0 \\ \varphi(\mathbf{x}), & \mathbf{x} \in \varkappa \backslash \varkappa_0 \end{cases}, \tag{A2.4}$$

where a is a parameter.

It is almost evident that

$$\lim_{a \to \infty} V_{\mathrm{loc}}\{\varphi_a\} = V_0. \tag{A2.5}$$

A2.2. Homogenization

In this case we transform $\varphi(\mathbf{x})$ into

$$\tilde{\varphi}_a(\mathbf{x}) \equiv \varphi(\mathbf{x}) + a.$$

Then $V_{\mathrm{loc}}\{\tilde{\varphi}_a\}$ is varied from $V_{\mathrm{loc}}\{\varphi\}$ to V, as a is varied from zero to infinity. This result is quite natural, since the greater the absolute value of a, the less is a relative difference of $\tilde{\varphi}_a(\mathbf{x})$ from the uniform distribution.

Now let us establish some more general property of $V_{\mathrm{loc}}\{\varphi\}$, expressed by the following lemma.

A2.3. Lemma

Suppose that

$$\frac{\sup_{\mathbf{x} \in \varkappa} |\varphi(\mathbf{x})|}{\langle |\varphi| \rangle} = \mu \geqslant 1. \tag{A2.6}$$

Then the localizational volume obeys the inequality

$$1 \leqslant \frac{V}{V_{\mathrm{loc}}\{\varphi\}} \leqslant \mu \Leftrightarrow \frac{1}{\mu} V \leqslant V_{\mathrm{loc}}\{\varphi\} \leqslant V. \tag{A2.7}$$

Before passing on to the proof, we note that the inequality (A2.7) means firstly, that if variation of $|\varphi(\mathbf{x})|$ with respect to its mean value is not great, then the localizational volume differs not greatly from V; secondly, if localizational volume is rather small relative to V, then $|\varphi(\mathbf{x})|$ for certain attains rather great relative values ($V/V_{\mathrm{loc}}\{\varphi\} \leqslant |\varphi(\mathbf{x})|/\langle |\varphi| \rangle \leqslant \mu$) on a set of $\varkappa_0$ of non-zero volume V_0 not exceeding $V_{\mathrm{loc}}\{\varphi\}$.

A2.3.1. Proof

Consider the following inequality:

$$\int_{\varkappa} \varphi^2 \, \mathrm{d}V = \int_{\varkappa} |\varphi| |\varphi| \, \mathrm{d}V \leqslant \sup_{\mathbf{x} \in \varkappa} |\varphi| \int_{\varkappa} |\varphi| \, \mathrm{d}V \tag{A2.8}$$

that is equivalent to

$$\langle \varphi^2 \rangle \leqslant \sup_{\mathbf{x}\in\varkappa} |\varphi| \langle |\varphi| \rangle = \frac{\sup_{\mathbf{x}\in\varkappa} |\varphi|}{\langle |\varphi| \rangle} \langle |\varphi| \rangle^2. \tag{A2.9}$$

Ultimately we have

$$\frac{V}{V_{\mathrm{loc}}\{\varphi\}} = \frac{\langle \varphi^2 \rangle}{\langle |\varphi| \rangle^2} \leqslant \frac{\sup_{\mathbf{x}\in\varkappa} |\varphi|}{\langle |\varphi| \rangle} = \mu. \tag{A2.10}$$

Note, that equality in eqn (A2.8), and hence, in eqn (A2.10), takes place if and only if $|\varphi(\mathbf{x})|$ takes only two values, specifically those of zero and

$$\sup_{\mathbf{x}\in\varkappa} |\varphi|,$$

the latter being taken on a subset $\varkappa_0$ of non-zero volume V_0. In this case $V_{\mathrm{loc}}\{\varphi\} = V_0$. Otherwise, i.e. if $|\varphi(\mathbf{x})|$ takes some values different from zero and

$$\sup_{\mathbf{x}\in\varkappa} |\varphi|,$$

then the inequality (A2.10) is strict and due to piecewise continuity of $\varphi(\mathbf{x})$ there is a subset $\varkappa_0$ of non-zero volume V_0 where the inequality

$$\frac{V}{V_{\mathrm{loc}}\{\varphi\}} \leqslant \frac{|\varphi(\mathbf{x})|}{\langle |\varphi| \rangle} \leqslant \mu \tag{A2.11}$$

is valid. Let us prove that $V_0 \leqslant V_{\mathrm{loc}}\{\varphi\}$. Integrating the first inequality in (A2.11) over $\varkappa_0$, we find:

$$\frac{V}{V_{\mathrm{loc}}\{\varphi\}} V_0 \leqslant \frac{1}{\langle |\varphi| \rangle} \int_{\varkappa_0} |\varphi| \, \mathrm{d}V \leqslant \frac{1}{\langle |\varphi| \rangle} \int_{\varkappa} |\varphi| \, \mathrm{d}V = V \Rightarrow V_0 \leqslant V_{\mathrm{loc}}\{\varphi\} \tag{A2.12}$$

Thus, the totality of declared properties of the quantity of localizational volume has been proved and it really evaluates the volume of that part of a region, where the scalar field takes relatively great absolute values.

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